Collatz Conjecture: Proposed Proof

Abstract

The Collatz Conjecture, also known as the 3n+1 problem, proposes the following:

> Given any natural number:
>
> - If odd: multiply by 3 and add 1
> - If even: divide by 2

Recursively applying this process to any natural number will result in a sequence which always returns to 1. We use the approach below to prove this conjecture.

Solution Approach:

The solution proposed to the Collatz Conjecture is divided into the following sections:

Section 1	We group odd numbers into branches, mapping each number to its parent root.
Section 2	Using the map constructed in Section 1, we apply modulo 3 to numbers in each branch to determine root types, establishing the foundational cyclical sequence 2,1,0 used for successive branch joining.
Section 3	We begin the process of joining child sub-branches derived from the seeding branch sequence described in Section 2, establishing the basis for a super-cycle which is further examine in Section 4.
Section 4	We deconstruct the super-cycle observed in Section 3 and analyze the underlying process for branch joining.
Section 5	We analyze the multi-dimensional hierarchical structure (graph) formed through successive branch joins, and introduce a recursive indexing process to preserve countability across dimensions.
Section 6	We construct a minimal subgraph from the cyclical sequences we examine, which is then expanded into an N-ary tree.
Section 7 (Conclusion)	We consider and subsequently reject the hypothesis that multiple independent graphs can exist, concluding that all numbers descend from branch root base 1.

Appendix:

Proofs for the assumptions and hypotheses proposed throughout this analysis are included as linked references to the appendix below.

- 1.1: F.1 -x = 1 (mod 3) Results in an integer
- 1.2: F.2 -x = 2 (mod 3) Results in an integer
- 2.1.1.1: F.1 -x = 1 (mod 3) Includes residuals 210 and cycles
- 2.1.1.2: F.2 -x = 2 (mod 3) Includes residuals 210 and cycles
- 2.2: F.1/F.2 Equivalent Sequences
- 2.3.1.1: F.PC.1.1 Includes residuals 210 and cycles
- 2.3.1.2: F.PC.1.2 Includes residuals 210 and cycles
- 2.3.2.1: F.PC.2.1 Includes residuals 210 and cycles
- 2.3.2.2: F.PC.2.2 Includes residuals 210 and cycles
- 3.1.1: F.1/F.2 Consistency for all Modulo 3^M
- 3.2: F.1/F.2 Equivalent Sequences for all Modulo 3^M
- 3.3.1: F.PC Consistency for all Modulo 3^M

Section 1: Branch Mapping (TOC)

We begin the proof below by considering the following characteristics of the conjecture's sequence:

1. Every odd number evaluates to an even number
2. Every even number is continuously divided by two until an odd number is reached

From these two observations, we can conclude that every odd number can be mapped to a different odd number, the only exception being the number 1 which cyclically resolves to itself.

The first step in this proof seeks to establish a method to map all odd numbers to their originating root number. We propose constructing this map using two exponential functions F.1 and F.2 defined below. We assign the applicable function to each odd number by using its additive inverse modulo 3. Using these two functions, we construct a method to arithmetically describe the branches derived from every odd number.

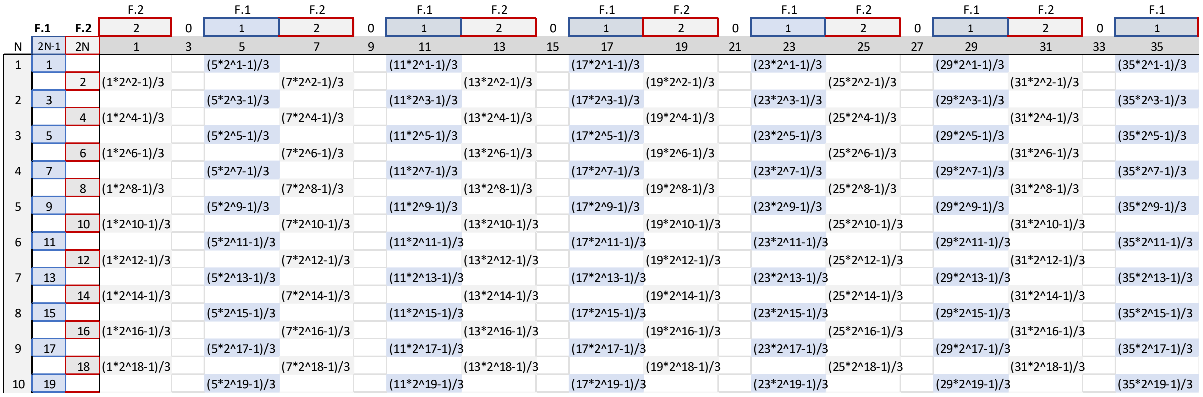
The three cases are defined as follows:

1. Case -X [Mod 3] =1: We use the function F.1 described as
$$\frac{x \cdot 2^{2(n-1)} - 1}{3}$$
2. Case -X [Mod 3] =2: We use the function F.2 described as
$$\frac{x \cdot 2^{2n} - 1}{3}$$
3. Case: -X [mod 3] = 0: This represents a case in which no even number can serve as a multiplier, such that subtracting one and dividing by three will result in a whole number.

Below we prove that the two functions F.1 and F.2 resolve to an integer for all natural numbers when applied to their respective case modulo 3.

1. 1.1: F.1 -x = 1 (mod 3) Results in an Integer
2. 2.1: F.2 -x = 2 (mod 3) Results in an Integer

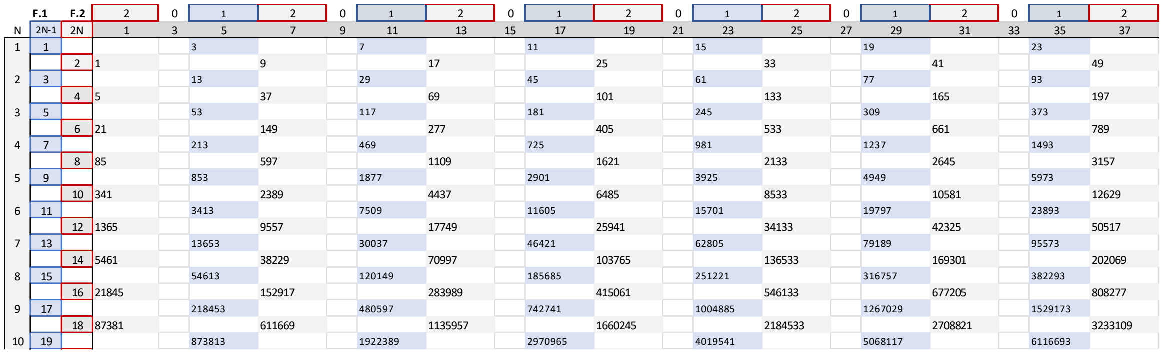
In Figure 1.1 we map each function F.1 and F.2 according to its respective case modulo 3.



			F.2		F.1	F.2		F.1	F.2		F.1	F.2		F.1	F.2		F.1	F.2		F.1
	F.1	F.2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35
1	1				(5*2^1-1)/3			(11*2^1-1)/3			(17*2^1-1)/3			(23*2^1-1)/3			(29*2^1-1)/3			(35*2^1-1)/3
		2	(1*2^2-1)/3			(7*2^2-1)/3			(13*2^2-1)/3			(19*2^2-1)/3			(25*2^2-1)/3			(31*2^2-1)/3		
2	3				(5*2^3-1)/3			(11*2^3-1)/3			(17*2^3-1)/3			(23*2^3-1)/3			(29*2^3-1)/3			(35*2^3-1)/3
		4	(1*2^4-1)/3			(7*2^4-1)/3			(13*2^4-1)/3			(19*2^4-1)/3			(25*2^4-1)/3			(31*2^4-1)/3		
3	5				(5*2^5-1)/3			(11*2^5-1)/3			(17*2^5-1)/3			(23*2^5-1)/3			(29*2^5-1)/3			(35*2^5-1)/3
		6	(1*2^6-1)/3			(7*2^6-1)/3			(13*2^6-1)/3			(19*2^6-1)/3			(25*2^6-1)/3			(31*2^6-1)/3		
4	7				(5*2^7-1)/3			(11*2^7-1)/3			(17*2^7-1)/3			(23*2^7-1)/3			(29*2^7-1)/3			(35*2^7-1)/3
		8	(1*2^8-1)/3			(7*2^8-1)/3			(13*2^8-1)/3			(19*2^8-1)/3			(25*2^8-1)/3			(31*2^8-1)/3		
5	9				(5*2^9-1)/3			(11*2^9-1)/3			(17*2^9-1)/3			(23*2^9-1)/3			(29*2^9-1)/3			(35*2^9-1)/3
		10	(1*2^10-1)/3			(7*2^10-1)/3			(13*2^10-1)/3			(19*2^10-1)/3			(25*2^10-1)/3			(31*2^10-1)/3		
6	11				(5*2^11-1)/3			(11*2^11-1)/3			(17*2^11-1)/3			(23*2^11-1)/3			(29*2^11-1)/3			(35*2^11-1)/3
		12	(1*2^12-1)/3			(7*2^12-1)/3			(13*2^12-1)/3			(19*2^12-1)/3			(25*2^12-1)/3			(31*2^12-1)/3		
7	13				(5*2^13-1)/3			(11*2^13-1)/3			(17*2^13-1)/3			(23*2^13-1)/3			(29*2^13-1)/3			(35*2^13-1)/3
		14	(1*2^14-1)/3			(7*2^14-1)/3			(13*2^14-1)/3			(19*2^14-1)/3			(25*2^14-1)/3			(31*2^14-1)/3		
8	15				(5*2^15-1)/3			(11*2^15-1)/3			(17*2^15-1)/3			(23*2^15-1)/3			(29*2^15-1)/3			(35*2^15-1)/3
		16	(1*2^16-1)/3			(7*2^16-1)/3			(13*2^16-1)/3			(19*2^16-1)/3			(25*2^16-1)/3			(31*2^16-1)/3		
9	17				(5*2^17-1)/3			(11*2^17-1)/3			(17*2^17-1)/3			(23*2^17-1)/3			(29*2^17-1)/3			(35*2^17-1)/3
		18	(1*2^18-1)/3			(7*2^18-1)/3			(13*2^18-1)/3			(19*2^18-1)/3			(25*2^18-1)/3			(31*2^18-1)/3		
10	19				(5*2^19-1)/3			(11*2^19-1)/3			(17*2^19-1)/3			(23*2^19-1)/3			(29*2^19-1)/3			(35*2^19-1)/3

Figure 1.1: Branch Function Mapping

In Figure 1.2, we evaluate the functions mapped in Figure 1.1.



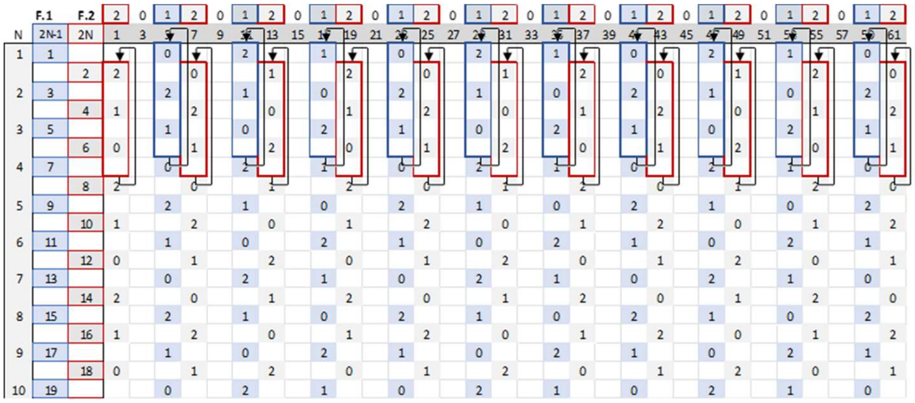
	F.1	F.2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37
1	1				3			7			11			15			19			23	
		2	1			9			17			25			33			41			49
2	3				13			29			45			61			77			93	
		4	5			37			69			101			133			165			197
3	5				53			117			181			245			309			373	
		6	21			149			277			405			533			661			789
4	7				213			469			725			981			1237			1493	
		8	85			597			1109			1621			2133			2645			3157
5	9				853			1877			2901			3925			4949			5973	
		10	341			2389			4437			6485			8533			10581			12629
6	11				3413			7509			11605			15701			19797			23893	
		12	1365			9557			17749			25941			34133			42325			50517
7	13				13653			30037			46421			62805			79189			95573	
		14	5461			38229			70997			103765			136533			169301			202069
8	15				54613			120149			185685			251221			316757			382293	
		16	21845			152917			283989			415061			546133			677205			808277
9	17				218453			480597			742741			1004885			1267029			1529173	
		18	87381			611669			1135957			1660245			2184533			2708821			3233109
10	19				873813			1922389			2970965			4019541			5068117			6116693	

Figure 1.2: Evaluated Numeric Mapping

We established above that every odd number resolves to a different odd number root, thus we can conclude the approach above presents a method to map the set of all odd numbers to their originating root, and consequently to a position in a corresponding parent branch. We also observe that every branch expands infinitely as we increment the variate N in the two functions we defined. These observations will be referenced later in the proof as we examine unbounded expansion as a constraint consideration.

Section 2: Mapped Branches Modulo 3 (TOC)

In Section 1 we used the modulated additive inverse of each number to identify its root branch type (1 or 2) and applied the relevant function F.1 or F.2 to derive its sequential branch. In Figure 2.1, we employ an equivalent method to each number (or node) within every branch, applying the additive inverse modulo 3. We observe this process results in the periodic sequence 2->1->0.



	F.1	F.2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61
1	1				0			2			1			0			2			1			0			2			1			0	
		2	2			0			1			2			0			1			2			0			1			2			0
2	3				2			1			0			2			1			0			2			1			0			2	
		4	1			2			0			1			2			0			1			2			0			1			2
3	5				1			0			2			1			0			2			1			0			2			1	
		6	0			1			2			0			1			2			0			1			2			0			1
4	7				0			2			1			0			2			1			0			2			1			0	
		8	2			0			1			2			0			1			2			0			1			2			0
5	9				2			1			0			2			1			0			2			1			0			2	
		10	1			2			0			1			2			0			1			2			0			1			2
6	11				1			0			2			1			0			2			1			0			2			1	
		12	0			1			2			0			1			2			0			1			2			0			1
7	13				0			2			1			0			2			1			0			2			1			0	
		14	2			0			1			2			0			1			2			0			1			2			0
8	15				2			1			0			2			1			0			2			1			0			2	
		16	1			2			0			1			2			0			1			2			0			1			2
9	17				1			0			2			1			0			2			1			0			2			1	
		18	0			1			2			0			1			2			0			1			2			0			1
10	19				0			2			1			0			2			1			0			2			1			0	

Figure 2.1: Modulated Cyclical Analysis: 2->1->0

In Figure 2.1 we observe the formation of a periodic cycle 2->1->0. This cycle will serve as a foundational sequence for generating recursive and successive branch joins. We seek to prove the following characteristics of the observed sequence are consistent and persist for both root types 1 and 2:

1. The sequence contains all residuals 2,1,0
2. The sequence is cyclical

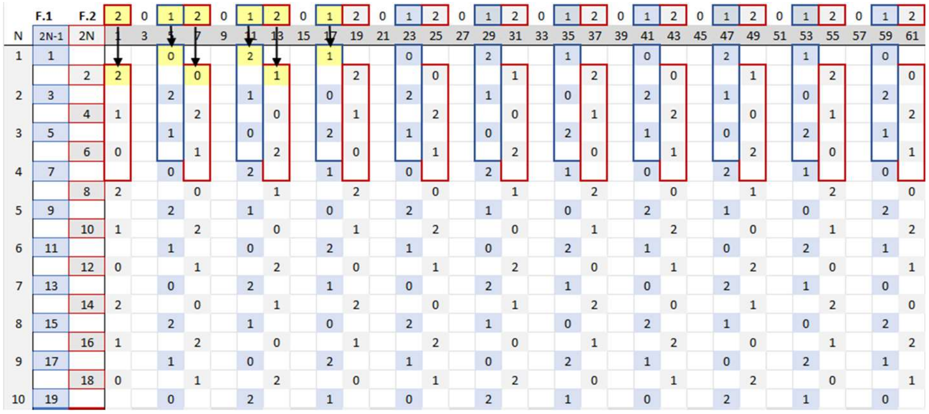
Below we use an inductive method to establish inclusive cyclic consistency.

- 2.1.1: F.1 -x = 1 (mod 3) Includes residuals 210 and cycles
- 2.1.1: F.2 -x = 2 (mod 3) Includes residuals 210 and cycles

We also seek to prove sequence order is equivalent (2-> 1-> 0) for both roots.

- 2.2: F.1/F.2 Equivalent Sequences

In Figure 2.2 below, we observe that each residual in the 2,1,0 cycle can serve as one of three child branch starting positions for both roots 1 and 2.



	F.1	F.2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61
1	1				0			2			1			0			2			1			0			2			1			0	
		2	2			0			1			2			0			1			2			0			1			2			0
2	3				2			1			0			2			1			0			2			1			0			2	
		4	1			2			0			1			2			0			1			2			0			1			2
3	5				1			0			2			1			0			2			1			0			2			1	
		6	0			1			2			0			1			2			0			1			2			0			1
4	7				0			2			1			0			2			1			0			2			1			0	
		8	2			0			1			2			0			1			2			0			1			2			0
5	9				2			1			0			2			1			0			2			1			0			2	
		10	1			2			0			1			2			0			1			2			0			1			2
6	11				1			0			2			1			0			2			1			0			2			1	
		12	0			1			2			0			1			2			0			1			2			0			1
7	13				0			2			1			0			2			1			0			2			1			0	
		14	2			0			1			2			0			1			2			0			1			2			0
8	15				2			1			0			2			1			0			2			1			0			2	
		16	1			2			0			1			2			0			1			2			0			1			2
9	17				1			0			2			1			0			2			1			0			2			1	
		18	0			1			2			0			1			2			0			1			2			0			1
10	19				0			2			1			0			2			1			0			2			1			0	

Figure 2.3: Equivalent Cycles for Branch Roots 1 and 2

We'll consider the hypothesis that in the context of modulo 3, we can use the root type and the child branch residual starting position to group all parent-child relationship pairs into six unique sets. In Section.3 we examine this further.

Section 3: Parent-Child Joins and Super-Cyclic Behavior (TOC)

In Section 1 we used functions F.1 and F.2 to map all odd numbers to their parental root. In Section 2 we reduced those sequences modulo 3, proving both periodicity and sequential equivalency. In Figure 2.3 we hypothesized that the residual starting position of each child branch, joined with its parent root type, can be used to establish six unique parent-child branch pairs.

The modulated map we constructed in Figure 2.3 however, examines a linear sequence constructed for the preliminary analysis and sequence grouping we examined. The branches derived using functions F.1 and F.2 both represent separate exponential sequences. To assess the hypothesis that a six pair set (modulo 3) is consistent for the sequences derived from F.1 and F.2, we need to examine the set of parent-child pairs derived from each branch. In Figure 3.1, we begin evaluating this by appending the starting number of each derived child branch to its corresponding parent number in the table we mapped in Figure 1.2.

			2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
1	1				3			7 -> 9			11 -> 7			15			19 -> 25	
		2	1 -> 1			9			17 -> 11			25 -> 33			33			41 -> 27
2	3				13 -> 17			29 -> 19			45			61 -> 81			77 -> 51	
		4	5 -> 3			37 -> 49			69			101 -> 67			133 -> 177			165
3	5				53 -> 35			117			181 -> 241			245 -> 163			309	
		6	21			149 -> 99			277 -> 369			405			533 -> 355			661 -> 881
4	7				213			469 -> 625			725 -> 483			981			1237 -> 1649	
		8	85 -> 113			597			1109 -> 739			1621 -> 2161			2133			2645 -> 1763
5	9				853 -> 1137			1877 -> 1251			2901			3925 -> 5233			4949 -> 3299	
		10	341 -> 227			2389 -> 3185			4437			6485 -> 4323			8533 -> 11377			10581
6	11				3413 -> 2275			7509			11605 -> 15473			15701 -> 10467			19797	
		12	1365			9557 -> 6371			17749 -> 23665			25941			34133 -> 22755			42325 -> 56433
7	13				13653			30037 -> 40049			46421 -> 30947			62805			79189 -> 105585	
		14	5461 -> 7281			38229			70997 -> 47331			103765 -> 13835			136533			169301 -> 11286
8	15				54613 -> 72817			120149 -> 80099			185685			251221 -> 33496			316757 -> 21117	

Figure 3.1: Numeric Parent -> Child Branch Map

In Figure 3.2, we next apply the same method of modulating the additive inverse to the parent-child set created in 3.1. We use this to examine the relationships derived from the exponential sequences generated by F.1 and F.2.

	F.1	F.2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49
1	1				0			2-0			1-2			0			2-2			1-0			0			2-1	
		2	2-2			0			1-1			2-0			0			1-0			2-1			0			1-2
2	3				2-1			1-2			0			2-0			1-0			0			2-2			1-1	
		4	1-0			2-2			0			1-2			2-0			0			1-1			2-1			0
3	5				1-1			0			2-2			1-2			0			2-1			1-0			0	
		6	0			1-0			2-0			0			1-2			2-1			0			1-1			2-2
4	7				0			2-2			1-0			0			2-1			1-1			0			2-0	
		8	2-1			0			1-2			2-2			0			1-1			2-0			0			1-0
5	9				2-0			1-0			0			2-2			1-1			0			2-1			1-2	
		10	1-1			2-1			0			1-0			2-2			0			1-2			2-0			0
6	11				1-2			0			2-1			1-0			0			2-0			1-1			0	
		12	0			1-1			2-2			0			1-0			2-0			0			1-2			2-1
7	13				0			2-1			1-1			0			2-0			1-2			0			2-2	
		14	2-0			0			1-0			2-1			0			1-2			2-2			0			1-1
8	15				2-2			1-1			0			2-1			1-2			0			2-0			1-0	
		16	1-2			2-0			0			1-1			2-1			0			1-0			2-2			0
9	17				1-0			0			2-0			1-1			0			2-2			1-2			0	
		18	0			1-2			2-1			0			1-1			2-2			0			1-0			2-0
10	19				0			2-0			1-2			0			2-2			1-0			0			2-1	
		20	2-2			0			1-1			2-0			0			1-0			2-1			0			1-2
11	21				2-1			1-2			0			2-0			1-0			0			2-2			1-1	
		22	1-0			2-2			0			1-2			2-0			0			1-1			2-1			0
12	23				1-1			0			2-2			1-2			0			2-1			1-0			0	
		24	0			1-0			2-0			0			1-2			2-1			0			1-1			2-2

Figure 3.2: Parent-Child Branch Map - Modulo3

In Figure 3.2, we observe a new super cyclical sequence with a period of nine, which also exhibits directional sequential equivalency for both root types 1 and 2.

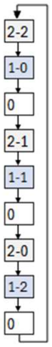
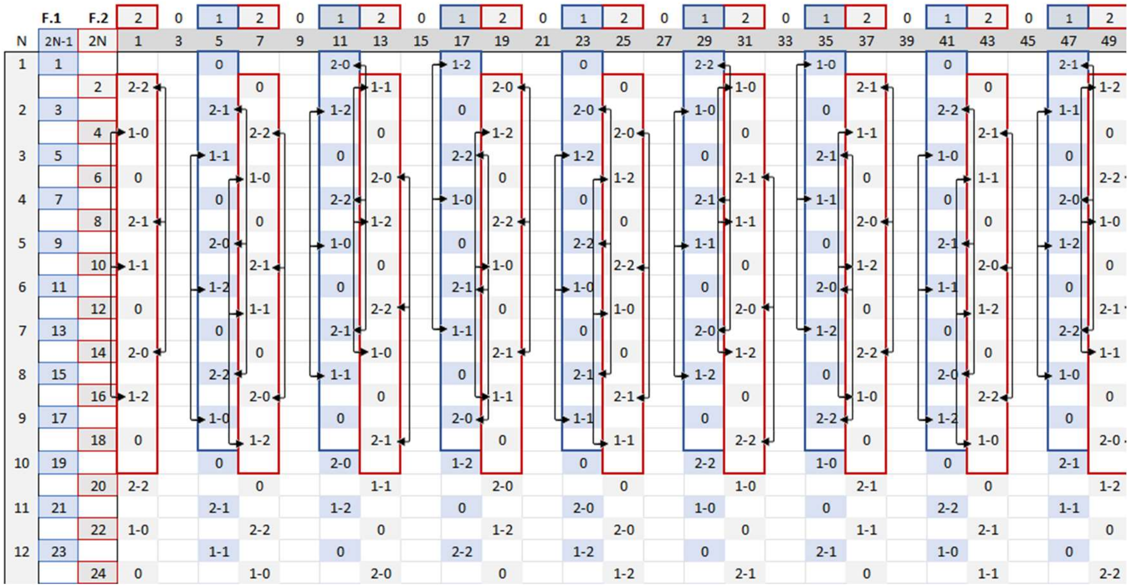


Figure 3.3: Parent-Child Branch Pair Super-Cycle



	F.1	F.2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49
1	1				0			2-0			1-2			0			2-2			1-0			0			2-1	
		2	2-2			0			1-1			2-0			0			1-0			2-1			0			1-2
2	3				2-1			1-2			0			2-0			1-0			0			2-2			1-1	
		4	1-0			2-2			0			1-2			2-0			0			1-1			2-1			0
3	5				1-1			0			2-2			1-2			0			2-1			1-0			0	
		6	0			1-0			2-0			0			1-2			2-1			0			1-1			2-2
4	7				0			2-2			1-0			0			2-1			1-1			0			2-0	
		8	2-1			0			1-2			2-2			0			1-1			2-0			0			1-0
5	9				2-0			1-0			0			2-2			1-1			0			2-1			1-2	
		10	1-1			2-1			0			1-0			2-2			0			1-2			2-0			0
6	11				1-2			0			2-1			1-0			0			2-0			1-1			0	
		12	0			1-1			2-2			0			1-0			2-0			0			1-2			2-1
7	13				0			2-1			1-1			0			2-0			1-2			0			2-2	
		14	2-0			0			1-0			2-1			0			1-2			2-2			0			1-1
8	15				2-2			1-1			0			2-1			1-2			0			2-0			1-0	
		16	1-2			2-0			0			1-1			2-1			0			1-0			2-2			0
9	17				1-0			0			2-0			1-1			0			2-2			1-2			0	
		18	0			1-2			2-1			0			1-1			2-2			0			1-0			2-0
10	19				0			2-0			1-2			0			2-2			1-0			0			2-1	
		20	2-2			0			1-1			2-0			0			1-0			2-1			0			1-2
11	21				2-1			1-2			0			2-0			1-0			0			2-2			1-1	
		22	1-0			2-2			0			1-2			2-0			0			1-1			2-1			0
12	23				1-1			0			2-2			1-2			0			2-1			1-0			0	
		24	0			1-0			2-0			0			1-2			2-1			0			1-1			2-2

Figure 4.4 Sub-Cycles within Each Branch Super-cycle

In Figure 4.4, we observe that each branch super-cycle can be further broken down into two separate sub-cycles. In Section 4, we analyze and break down the generative process underlying the formation of this sequence.

Section 4: Periodic Super-Cyclical Decomposition and Analysis (TOC)

In Section 2 we examined linear branch sequences modulo 3, proving both periodicity and equivalent directional ordering for branches emanating from root types 1 and 2 (seq. 2->1->0). This cycle will be used as a basis for successive generative sequencing, and later for subsequent branch join indexing.

We considered the hypothesis that joins between parental roots and their successive child branch residuals can be bounded to a finite cyclical sequence as we observed in Figure 3.3. We constructed a nine period cyclical sequence through parent-child unions derived from exponential functions F.1. and F.2. in the context of modulo 3. We deconstruct this generative process in Figure 4.1 below.

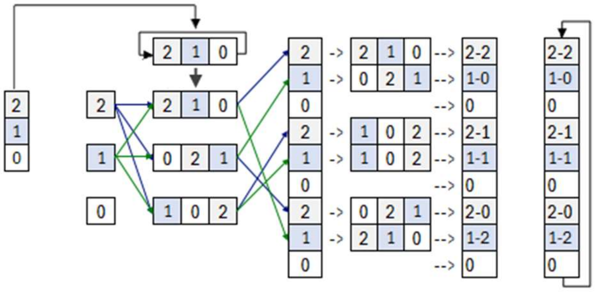


Figure 4.1: Parent-Child Generative Process Case Modulo 3

In Figure 4.1, we propose a process for joining parent-child branch pairs and subsequently transforming them into a linear cyclical sequence. In the figure above, we distinguish the parent-child joins emanating from root types 1 and 2 (within a given branch) into separate cycles, both possessing the same period and residual set, but generating independent cyclical sequences.

Root Type 1: 2->0->1
Root Type 2: 2->1->0

We then join these two sequential cycles with the 2->1->0 seed root cycle to construct the linear super-cycle observed in Figure 3.3.

The hypothesis we considered above proposes that the resulting super-cycle is both periodic and inclusive of all residuals for both branch root types 1 and 2. To prove this, we examine four cases below:

1. Case 1: Branch is of root type 1
    - 1.1. Nodes within the branch of root type 1 produce a cyclical sequence including all residuals.
    - 1.2. Nodes within the branch of root type 2 produce a cyclical sequence including all residuals

2. Case 2: Branch is of root type 2
    - 2.1. Nodes within the branch of root type 1 produce a cyclical sequence including all residuals
    - 2.2. Nodes within the branch of root type 2 produce a cyclical sequence including all residuals

We use an inductive method in the four proofs below to validate the predicates above:

- 4.1.1: F.PC.1.1 Includes residuals 210 and cycles
- 4.1.2: F.PC.1.2 Includes residuals 210 and cycles
- 4.2.1: F.PC.2.1 Includes residuals 210 and cycles
- 4.2.2: F.PC.2.2 Includes residuals 210 and cycles

In Section 5, we examine the multi-dimensional structure (graph) formed as branches are successively joined.

Section 5: Branch Joining and Cross-Dimensional Indexing (TOC)

In Section 4 we examined the process of joining child branches (modulo 3) with the foundational branch seeding sequence 2->1->0. As branches are successively joined, the resulting structure evolves into an unbounded multi-dimensional hierarchical graph. Given that this structure is constructed using functions F.1 and F.2, which represent exponential expressions, we can assume the graph is also inherently directional.

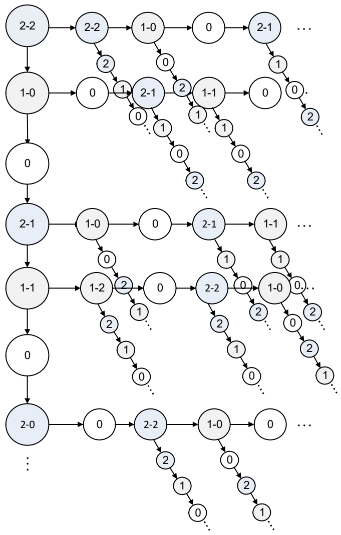
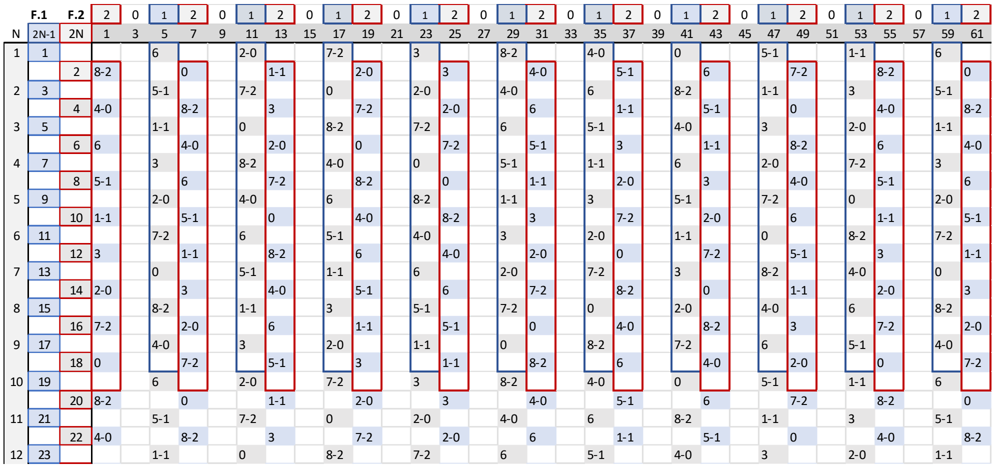


Figure 5.1 Cross-Dimensional Branch Joining Modulo 3

In Section 4 we examined a sequence to describe basic parent-child branch joins, producing a bounded cyclical sequence that is thus countable. In Figure 5.1 we examined the formation of an unbounded hierarchical structure as branches are successively joined. Next, we seek to establish a method to preserve countability across dimensions as they are constructed through continuous recursive branch joining. In Figure 5.2 we examine the concept of sequence indexing by increasing the parent to modulo 9, allowing us to uniquely define the sequence.



	F.1	F.2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61
1	1				6			2-0			7-2			3			8-2			4-0			0			5-1			1-1			6	
		2	8-2			0			1-1			2-0			3			4-0			5-1			6			7-2			8-2			0
2	3				5-1			7-2			0			2-0			4-0			6			8-2			1-1			3			5-1	
		4	4-0			8-2			3			7-2			2-0			6			1-1			5-1			0			4-0			8-2
3	5				1-1			0			8-2			7-2			6			5-1			4-0			3			2-0			1-1	
		6	6			4-0			2-0			0			7-2			5-1			3			1-1			8-2			6			4-0
4	7				3			8-2			4-0			0			5-1			1-1			6			2-0			7-2			3	
		8	5-1			6			7-2			8-2			0			1-1			2-0			3			4-0			5-1			6
5	9				2-0			4-0			6			8-2			1-1			3			5-1			7-2			0			2-0	
		10	1-1			5-1			0			4-0			8-2			3			7-2			2-0			6			1-1			5-1
6	11				7-2			6			5-1			4-0			3			2-0			1-1			0			8-2			7-2	
		12	3			1-1			8-2			6			4-0			2-0			0			7-2			5-1			3			1-1
7	13				0			5-1			1-1			6			2-0			7-2			3			8-2			4-0			0	
		14	2-0			3			4-0			5-1			6			7-2			8-2			0			1-1			2-0			3
8	15				8-2			1-1			3			5-1			7-2			0			2-0			4-0			6			8-2	
		16	7-2			2-0			6			1-1			5-1			0			4-0			8-2			3			7-2			2-0
9	17				4-0			3			2-0			1-1			0			8-2			7-2			6			5-1			4-0	
		18	0			7-2			5-1			3			1-1			8-2			6			4-0			2-0			0			7-2
10	19				6			2-0			7-2			3			8-2			4-0			0			5-1			1-1			6	
		20	8-2			0			1-1			2-0			3			4-0			5-1			6			7-2			8-2			0
11	21				5-1			7-2			0			2-0			4-0			6			8-2			1-1			3			5-1	
		22	4-0			8-2			3			7-2			2-0			6			1-1			5-1			0			4-0			8-2
12	23				1-1			0			8-2			7-2			6			5-1			4-0			3			2-0			1-1	

Figure 5.2: Parent Modulo 9 / Child Modulo 3

In Figure 5.2 we observe the ordered cycle sets period nine persists for both branch types 1 and 2.

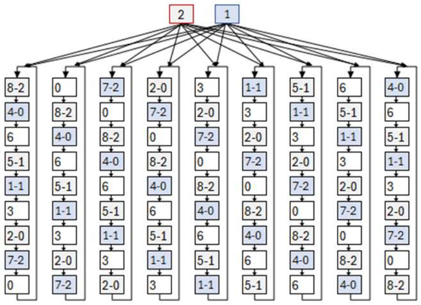
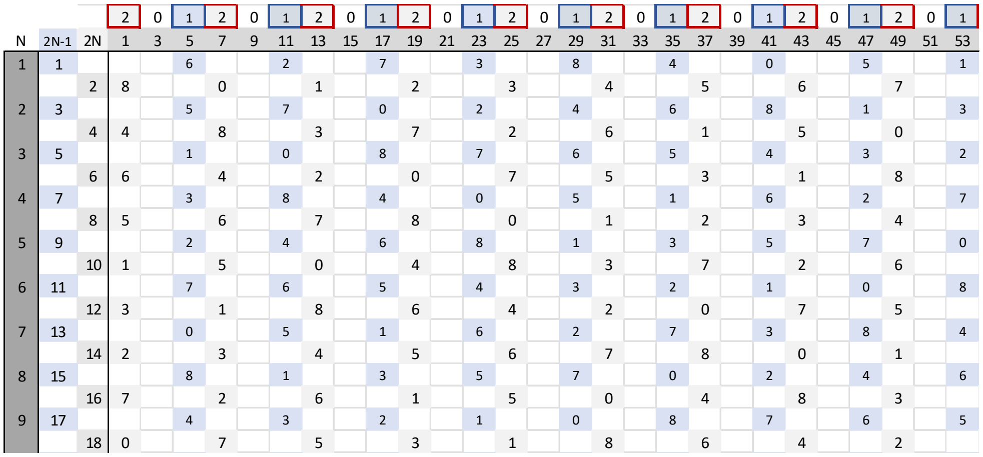


Figure 5.3: Parent Modulo 9 Parent / Child Modulo 3 Pairs for Both Root Branche Types 1 and 2

Given that we've uniquely mapped the sequence modulo 9 using each parent node, we can reduce the map in 5.3 to its parent modulo 9 indexing value.



			2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53
1	1				6			2			7			3			8			4			0			5			1
		2	8			0			1			2			3			4			5			6			7		
2	3				5			7			0			2			4			6			8			1			3
		4	4			8			3			7			2			6			1			5			0		
3	5				1			0			8			7			6			5			4			3			2
		6	6			4			2			0			7			5			3			1			8		
4	7				3			8			4			0			5			1			6			2			7
		8	5			6			7			8			0			1			2			3			4		
5	9				2			4			6			8			1			3			5			7			0
		10	1			5			0			4			8			3			7			2			6		
6	11				7			6			5			4			3			2			1			0			8
		12	3			1			8			6			4			2			0			7			5		
7	13				0			5			1			6			2			7			3			8			4
		14	2			3			4			5			6			7			8			0			1		
8	15				8			1			3			5			7			0			2			4			6
		16	7			2			6			1			5			0			4			8			3		
9	17				4			3			2			1			0			8			7			6			5
		18	0			7			5			3			1			8			6			4			2		

Figure 5.4: Sequence F.1 and F.2 Parent Modulo 9

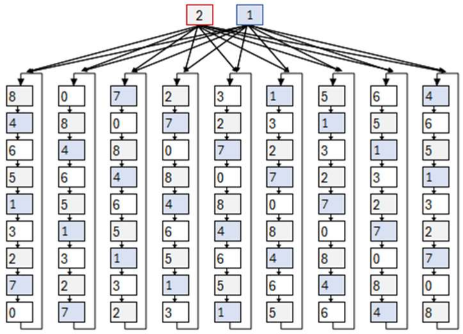


Figure 5.5: Modulo 9 Repeating Indexed Sequence Cycle for Branch Types 1 and 2

In Figure 5.6, we next examine applying the same generative process introduced in Section 4. We join each residual value in the modulo 9 child branch cycle to roots 1 and 2 in the root seeding sequence 2->1->0. After joining these sets into its linear cyclical form, we next employ the same indexing approach used for case modulo 3.

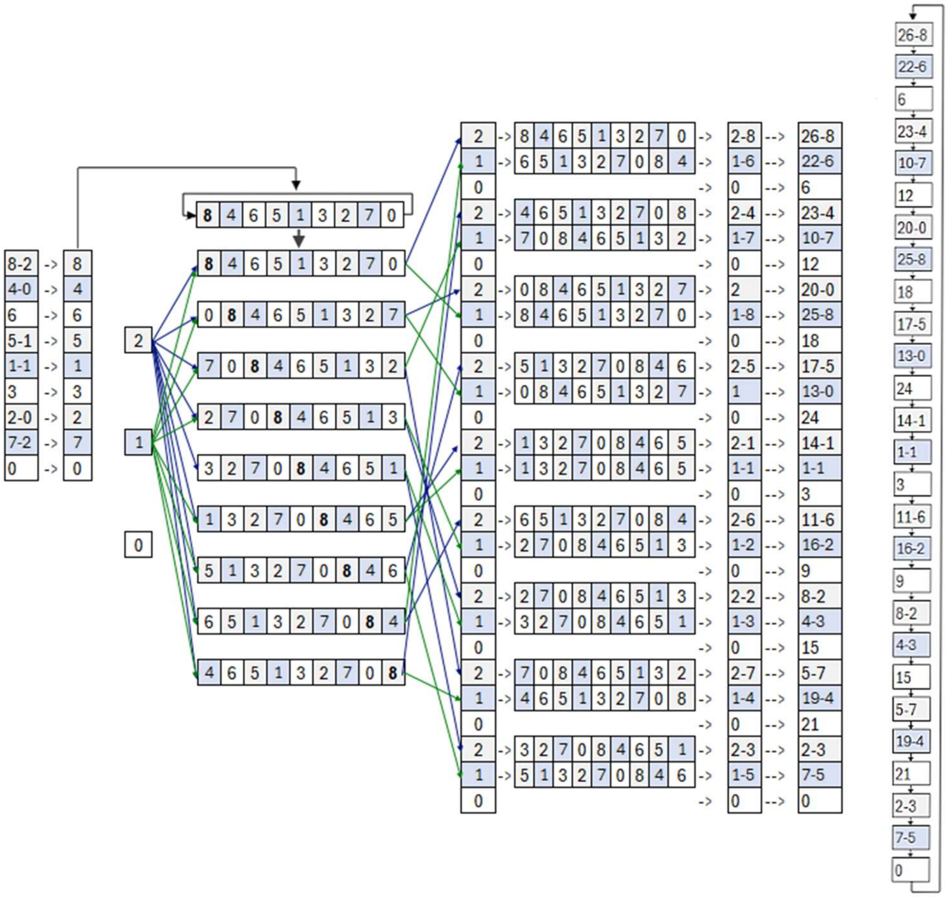
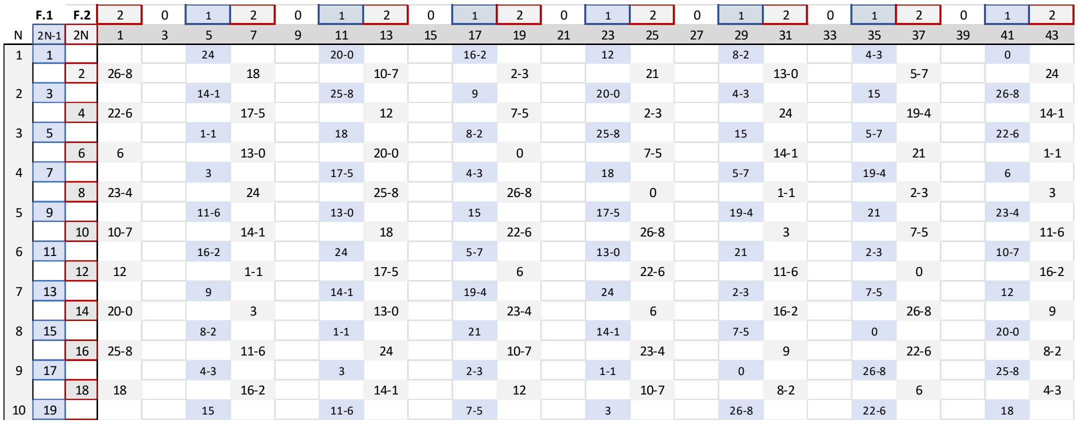


Figure 5.6: Iterative Child/Parent Generative Process Using Sequence Modulo 9

In Figure 5.6, we observe a new super-cycle with a period of 27. In Figure 5.7, we construct the table using parent modulo 27 and child modulo 9 respectively to analyze and verify cyclical persistence.



	F.1	F.2	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
N	2N-1	2N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43
1	1				24			20-0			16-2			12			8-2			4-3			0	
		2	26-8			18			10-7			2-3			21			13-0			5-7			24
2	3				14-1			25-8			9			20-0			4-3			15			26-8	
		4	22-6			17-5			12			7-5			2-3			24			19-4			14-1
3	5				1-1			18			8-2			25-8			15			5-7			22-6	
		6	6			13-0			20-0			0			7-5			14-1			21			1-1
4	7				3			17-5			4-3			18			5-7			19-4			6	
		8	23-4			24			25-8			26-8			0			1-1			2-3			3
5	9				11-6			13-0			15			17-5			19-4			21			23-4	
		10	10-7			14-1			18			22-6			26-8			3			7-5			11-6
6	11				16-2			24			5-7			13-0			21			2-3			10-7	
		12	12			1-1			17-5			6			22-6			11-6			0			16-2
7	13				9			14-1			19-4			24			2-3			7-5			12	
		14	20-0			3			13-0			23-4			6			16-2			26-8			9
8	15				8-2			1-1			21			14-1			7-5			0			20-0	
		16	25-8			11-6			24			10-7			23-4			9			22-6			8-2
9	17				4-3			3			2-3			1-1			0			26-8			25-8	
		18	18			16-2			14-1			12			10-7			8-2			6			4-3
10	19				15			11-6			7-5			3			26-8			22-6			18	

Figure 5.7: Modulo 27 Parent / Modulo 9 Child Pairs

In the process described above, we proposed a method to preserve countability through successive branch joins. In Figure 5.7, we demonstrate the application of this method to preserve countability sets across dimensions.

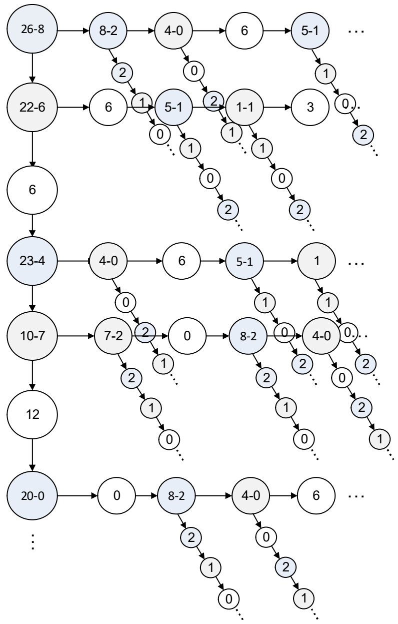


Figure 5.8: Graph Construct Using Recursively Reduced Indexed Mapping

In the sample cases we analyzed, the cardinality of each indexing set was increased by a factor of three to uniquely define the sequence. Given the process we examined joins the seeding cycle 2->1->0, which has a set cardinality three, with each cycle's previous residual set, we hypothesize that scaling each successive index set size by a factor of three will consistently yield equivalently countable sequence maps.

To validate this hypothesis, we first consider that our base case modulo 3 demonstrated the following conditions were true:

1. Branch sequences contained all residuals and were cyclical.
2. Branch types 1 and 2 both exhibited equivalent cyclical directionality.
3. We then separated cycles 1 and 2 within each branch, establishing independent but consistent cyclical sequences. After merging these cycles and indexing the resulting sequence, we observed new cyclical and directional equivalent branches for both branch root types (1 and 2).

To confirm the process will consistently exhibit equivalent behavior as we increment the modulus base M, we need to show the following hold for all cases 3^^{M}.

1. Sequences F.1 and F.2 remain cyclical and inclusive of all residuals (*1)
2. Sequences F.1 and F.2 maintain directional sequence equivalency
3. Parent-child unions remain cyclical and inclusive of all residuals (*2)

We use Hensel's lemma below to lift the sequences we previously proved in the context of modulo 3.

1. 5.1.1: F.1/F.2 Consistency for all Modulo 3^^{M} (*1)
2. 5.2: F.1/F.2 Equivalent Sequences for all Modulo 3^^{M}
3. 5.3.1: F.PC Consistency for all Modulo 3^^{M} (*2)

(*1) In Section 2 we included separate proofs for functions F.1 and F.2 to establish a 2,1,0 cyclical sequence for both root types 1 and 2. Since the sequence and structure of these functions do not fundamentally differ, we limit our proof for case 3^^{M} to F.1 to limit redundancy since it can be reasonably assumed this would equivalently hold for F.2.

(*2) In Section 4, parent/branch joins were divided into four separate proofs (4.1.1, 4.1.2, 4.2.1, 4.2.2). We limit our proof for all 3^^{M} to the first case 4.1.1. Since the four cases do not fundamentally differ in their underlying mathematical properties, we can assume the proof equivalently holds for the remaining cases.

Section 6: Minimal Subgraph Mapping and N-ary Tree Expansion (TOC)

In Section 5, we established that a directed N-dimensional hierarchical structure is constructed as we successively join branches. We demonstrated a recursive method of indexing parent-child relationships to preserve countability as branches are joined into new dimensional spaces. Given that sets are cyclical and ordered in each dimension, cross-dimensional joins can be mapped using a minimum sub-set of the parent-child pairs we examined.

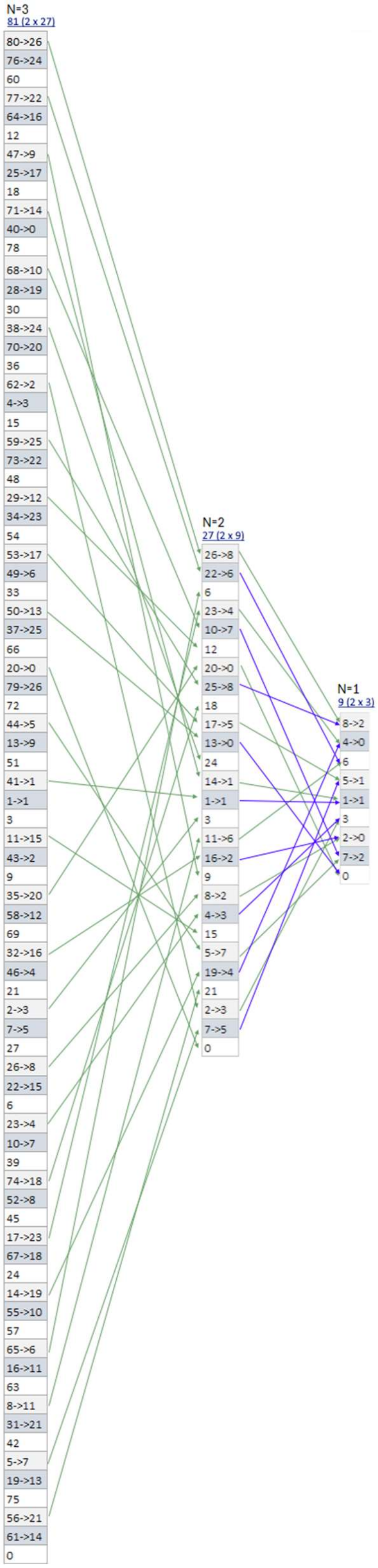


Figure 6.1: Example of Minimal Subgraph Cross-Dimensional Joining

In Figure 6.1, we used a minimal subgraph to demonstrate basic cross dimensional joining. To express a complete set of unique parent-child combinations, we can expand the subgraph from 6.1 into an N-ary tree using the cyclical sequence at each level.

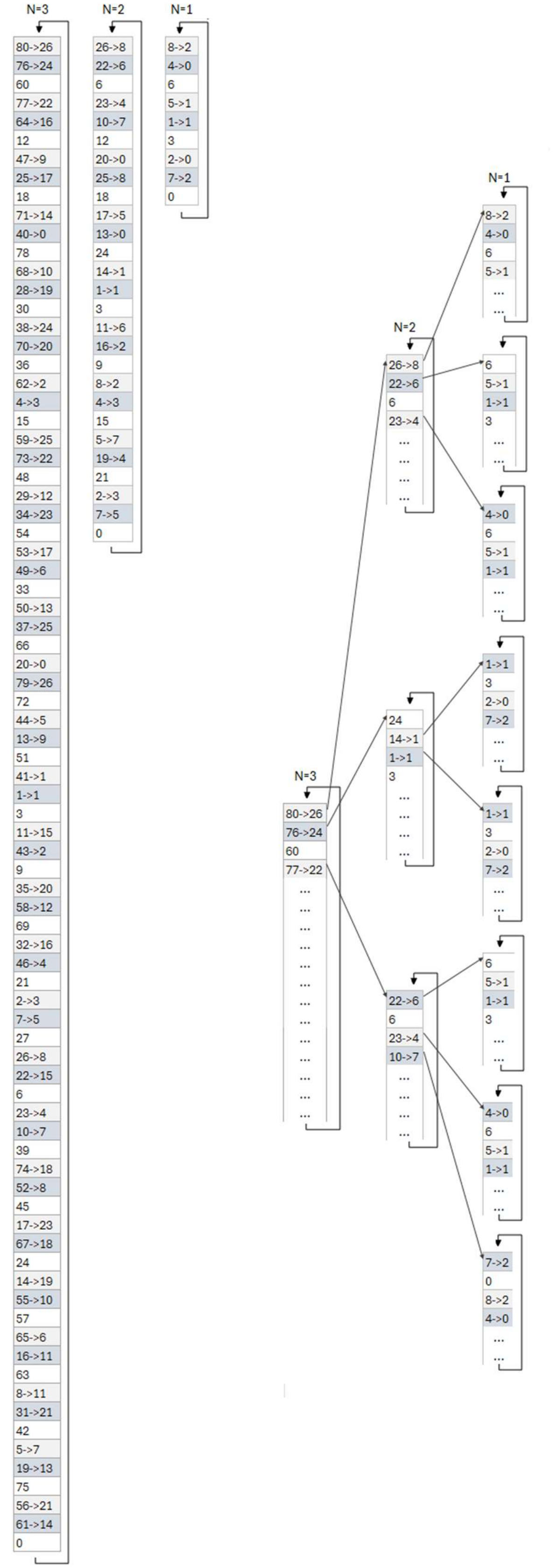


Figure 6.2: N-ary Tree Modeling

In Figure 6.2, we expand the minimal subgraph map constructed in Figure 6.1 to represent a fully expressive N–ary tree using the cycle maps for each level.

Next, we consider the following properties of the resulting tree to determine its branching growth factor:

1. Branches at each level have 3^$^{(N+1)}$ nodes
2. The set of nodes in a parent branch is 3 times greater than their descendent sub-branches [3^$^{(N+2)}$]
3. 2/3 of the nodes in each parent seeding branch produce sub-branches

Considering these properties, we can calculate the ratio of nodes in a parent branch relative to its successive sub-branch as:

$$\frac{3^{(N+2)}}{3^{(N+1)} \cdot \left(3^{(N+2)} \cdot \frac{2}{3}\right)}$$

This simplifies to:

$$\frac{1}{\left(3^{(N+1)}\right) \cdot \frac{2}{3}}$$

This ratio represents a diminishing sequence, demonstrating that the number of nodes in a parent branch is smaller than the sum of the nodes in its collective sub-branches. This allows us to conclude that every graph generated from the recursive application of F.1 and F.2 demonstrates strict growth.

Section 7: Conclusion (TOC)

We demonstrated that recursively applying functions F.1 and F.2 to any branch forms an unbounded hierarchical graph.

We consider the hypothesis that multiple independent graphs can exist that satisfy the following conditions:

1. **Bi-infinitcy:**
   In Section 1, we demonstrated that all branches derive from a root node in a parent branch. Each branch also yields an infinite number of sub-branches. From this we conclude that all branches, with the exception of the branch root base 1, are inherently bi-infinte.
2. **Hierarchical Directionality:**
   In Section 5, we demonstrated that branch joins result in a hierarchical structure. Given that the generative mapping functions F.1 and F.2 are exponential, we can also conclude directionality.
3. **Acyclicality and Relative Convergence:**
   In Section 6, we modeled pair sets into a tree, establishing that each level in the tree progressively converges, which also eliminates the possibility of cycles.

From conditions 2 and 3, we can conclude that all structures formed by successive branch joins constitute a directed acyclic graph with relative convergence, which cannot be unbounded in both directions. Conditions 1, 2 and 3 are thus inconsistent, allowing us to reject the hypothesis that multiple independent graphs can exist. We therefore conclude that all numbers are descendants of the graph emanating from branch root 1.

Appendix: Proofs (TOC)

- 1.1: F.1 -x = 1 (mod 3) Results in an Integer
- 1.2: F.2 -x = 2 (mod 3) Results in an Integer
- 2.1.1: F.1 -x = 1 (mod 3) Includes residuals 210 and cycles
- 2.1.2: F.2 -x = 2 (mod 3) Includes residuals 210 and cycles
- 2.2: F.1/F.2 Equivalent Sequences
- 4.1.1: F.PC.1.1 Includes residuals 210 and cycles
- 4.1.2: F.PC.1.2 Includes residuals 210 and cycles
- 4.2.1: F.PC.2.1 Includes residuals 210 and cycles
- 4.2.2: F.PC.2.2 Includes residuals 210 and cycles
- 5.1.1: F.1/F.2 Consistency for all Modulo 3^M
- 5.2: F.1/F.2 Equivalent Sequences for all Modulo 3^M
- 5.3.1: F.PC Consistency for all Modulo 3^M

1.1: F.1 -x = 1 (mod 3) results in an Integer (appendix)

Proof

We seek to prove that if $-x \equiv 1 \pmod 3$, then the expression

$$\frac{x \cdot 2^{2(n-1)} - 1}{3}$$

is an integer for all positive integers n.

Step 1: Simplifying the Condition $-x \equiv 1 \pmod 3$

Given $-x \equiv 1 \pmod 3$, we have:

$$x \equiv 2 \pmod 3.$$

Step 2: Substituting $x \equiv 2 \pmod 3$ into the Expression

Substitute $x \equiv 2 \pmod 3$ into the expression:

$$\frac{x \cdot 2^{2(n-1)} - 1}{3}.$$

This gives:

$$\frac{2 \cdot 2^{2(n-1)} - 1}{3}.$$

Step 3: Simplifying the Numerator

Notice that $2^2 = 4 \equiv 1 \pmod 3$, so $2^{2(n-1)} \equiv 1 \pmod 3$. Thus,

$$2 \cdot 2^{2(n-1)} \equiv 2 \cdot 1 \equiv 2 \pmod 3.$$

Therefore, the expression becomes:

$$\frac{2 \cdot 2^{2(n-1)} - 1}{3} \equiv \frac{2-1}{3} = \frac{1}{3} \pmod 3.$$

However, since $2 - 1 \equiv 1 \pmod 3$, and the original expression

$$\frac{2 \cdot 2^{2(n-1)} - 1}{3}$$

is divisible by 3 for all n.

Step 4: Conclusion

Since the numerator $2 \cdot 2^{2(n-1)} - 1$ is divisible by 3 for all n, it follows that

$$\frac{x \cdot 2^{2(n-1)} - 1}{3}$$

is an integer for all positive integers n.

1.2: F.2 -x = 2 (mod 3) Results in an Integer (appendix)

Proof

We seek to prove that if $-x \equiv 2 \pmod 3$, then the expression

$$\frac{x \cdot 2^{2n} - 1}{3}$$

is an integer for all positive integers n.

Step 1: Simplifying the Condition $-x \equiv 2 \pmod 3$

Given $-x \equiv 2 \pmod 3$, we have:

$$x \equiv 1 \pmod 3.$$

Step 2: Substituting $x \equiv 1 \pmod 3$ into the Expression

Substitute $x \equiv 1 \pmod 3$ into the expression:

$$\frac{x \cdot 2^{2n} - 1}{3}.$$

This gives:

$$\frac{1 \cdot 2^{2n} - 1}{3} = \frac{2^{2n} - 1}{3}.$$

Step 3: Simplifying the Numerator

Notice that $2^{2n} \equiv 1 \pmod 3$ for all n, because $2^2 = 4 \equiv 1 \pmod 3$, and so $2^{2n} \equiv 1^n \equiv 1 \pmod 3$.

Therefore, the expression becomes:

$$\frac{2^{2n} - 1}{3} \equiv \frac{1 - 1}{3} = \frac{0}{3} = 0 \pmod 3.$$

Step 4: Conclusion

Since the numerator $2^{2n} - 1$ is divisible by 3 for all n, it follows that

$$\frac{x \cdot 2^{2n} - 1}{3}$$

is an integer for all positive integers n.

2.1.1_M3_EB_RS_CS_1 **(appendix)**

Proof of Residuals and Single Cycle for Function $h(x, n)$

Given the function:

$$h(x, n) = -\left(\frac{x \cdot 2^{2n-1}}{3}\right) \mod 3.$$

We seek to prove that this function includes all residues $2, 1, 0 \mod 3$ and has a single repeating cycle.

Step 1: Analyze Powers of 2 Modulo 3

The powers of 2 modulo 3 cycle with a period of 2:

$2^1 \equiv 2 \pmod 3, \quad 2^2 \equiv 1 \pmod 3, \quad 2^3 \equiv 2 \pmod 3, \quad 2^4 \equiv 1 \pmod 3, \ldots$

Thus, for 2^{2n-1}, the powers alternate as:

- If n is odd: $2^{2n-1} \equiv 2 \pmod 3$,
- If n is even: $2^{2n-1} \equiv 1 \pmod 3$.

Step 2: Simplify the Function $h(x, n) \mod 3$

We start with the expression for $h(x, n)$:

$$h(x, n) = -\left(\frac{x \cdot 2^{2n-1}}{3}\right) \mod 3.$$

Substitute $x \mod 3$ values (for $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 3$, or $x \equiv 0 \pmod 3$) and simplify for both odd and even values of n.

Case 1: Odd n

When n is odd:

$$2^{2n-1} \equiv 2 \pmod 3.$$

Thus, for odd n, we have:

$$h(x, n) = -\left(\frac{x \cdot 2}{3}\right) \mod 3.$$

Substituting $x \equiv 1 \pmod 3$, we get:

$$h(x, n) = -\left(\frac{2}{3}\right) \equiv 1 \pmod 3.$$

Similarly, if $x \equiv 2 \pmod 3$, we get:

$$h(x, n) = -\left(\frac{4}{3}\right) \equiv 2 \pmod 3.$$

Case 2: Even n

When n is even:

$$2^{2n-1} \equiv 1 \pmod 3.$$

Thus, for even n, we have:

$$h(x, n) = -\left(\frac{x \cdot 1}{3}\right) \mod 3.$$

Substituting $x \equiv 1 \pmod 3$, we get:

$$h(x, n) = -\left(\frac{1}{3}\right) \equiv 2 \pmod 3.$$

Similarly, if $x \equiv 2 \pmod 3$, we get:

$$h(x, n) = -\left(\frac{2}{3}\right) \equiv 1 \pmod 3.$$

Step 3: Residue Analysis

By alternating between even and odd values of n, the function $h(x, n) \mod 3$ produces different outputs based on the alternating behavior of $2^{2n-1} \mod 3$. Specifically:

- For odd n, $h(x, n) \equiv 2 \pmod 3$,
- For even n, $h(x, n) \equiv 1 \pmod 3$.

Additionally, the function may produce the residue $0 \pmod 3$ in certain cases where the value inside the expression becomes divisible by 3. This can happen when the factor $x \cdot 2^{2n-1}$ is a multiple of 3.

Thus, $h(x, n) \mod 3$ cycles through the residues $2, 1, 0$.

Step 4: Proving a Single Cycle

- **Finite Set of Residues:** Since we are working modulo 3, the possible residues are restricted to $2, 1, 0$. This means there is a finite set of outcomes.
- **Injectivity and Surjectivity:** We can see that the function maps n to unique values for both odd and even cases. Every possible value (2, 1, 0) is covered, so the function is surjective. Moreover, since the values map uniquely without branching into different outputs, the function is injective.
- **Periodic Behavior:** Given that we have a finite number of possible values (3 residues), injectivity, and surjectivity, the function must repeat after a fixed number of steps, forming a cycle that includes all residues.

Conclusion

The function $h(x, n) \mod 3$ includes all residues $2, 1, 0 \mod 3$, and since it is injective and surjective over this finite set of values, it must form a **single repeating cycle** that covers all these residues.

Proof of Residuals and Single Cycle for Function $h(x, n)$

We are given the function:

$$h(x,n) = \left[-\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right)\right] \mod 3.$$

We seek to prove that this function includes all residues $2, 1, 0 \mod 3$ and has a single repeating cycle.

Step 1: Analyze Powers of 2 Modulo 3

The powers of 2 modulo 3 cycle with a period of 2:

$$2^1 \equiv 2 \pmod 3, \quad 2^2 \equiv 1 \pmod 3, \quad 2^3 \equiv 2 \pmod 3, \quad 2^4 \equiv 1 \pmod 3, \ldots$$

Thus, for $2^{2(n-1)}$, the powers alternate as:

- If n is odd: $2^{2(n-1)} \equiv 1 \pmod 3$,
- If n is even: $2^{2(n-1)} \equiv 2 \pmod 3$.

Step 2: Simplify the Function $h(x, n) \mod 3$

We start with the expression for $h(x, n)$:

$$h(x,n) = \left[-\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right)\right] \mod 3.$$

Substitute $x \mod 3$ values (for $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 3$, or $x \equiv 0 \pmod 3$) and simplify for both odd and even values of n.

Case 1: Odd n

When n is odd:

$$2^{2(n-1)} \equiv 1 \pmod 3.$$

Thus, for odd n, we have:

$$h(x,n) = \left[-\left(\frac{x \cdot 1 - 1}{3}\right)\right] \mod 3.$$

Substituting $x \equiv 1 \pmod 3$, we get:

$$h(x,n) = \left[-\left(\frac{1-1}{3}\right)\right] \equiv 0 \pmod 3.$$

Similarly, if $x \equiv 2 \pmod 3$, we get:

$$h(x,n) = \left[-\left(\frac{2-1}{3}\right)\right] \equiv 2 \pmod 3.$$

Case 2: Even n

When n is even:

$$2^{2(n-1)} \equiv 2 \pmod 3.$$

Thus, for even n, we have:

$$h(x,n) = \left[-\left(\frac{x \cdot 2 - 1}{3}\right)\right] \mod 3.$$

Substituting $x \equiv 1 \pmod 3$, we get:

$$h(x,n) = \left[-\left(\frac{2-1}{3}\right)\right] \equiv 2 \pmod 3.$$

Similarly, if $x \equiv 2 \pmod 3$, we get:

$$h(x,n) = \left[-\left(\frac{4-1}{3}\right)\right] \equiv 1 \pmod 3.$$

Step 3: Residue Analysis

By alternating between even and odd values of n, the function $h(x,n) \mod 3$ produces different outputs based on the alternating behavior of $2^{2(n-1)} \mod 3$. Specifically:

- For odd n, $h(x,n) \equiv 0 \pmod 3$ or $2 \pmod 3$,
- For even n, $h(x,n) \equiv 1 \pmod 3$ or $2 \pmod 3$.

Thus, $h(x,n) \mod 3$ cycles through the residues $2, 1, 0$.

Step 4: Proving a Single Cycle

- **Finite Set of Residues:** Since we are working modulo 3, the possible residues are restricted to $2, 1, 0$. This means there is a finite set of outcomes.
- **Injectivity and Surjectivity:** We can see that the function maps n to unique values for both odd and even cases. Every possible value (2, 1, 0) is covered, so the function is surjective. Moreover, since the values map uniquely without branching into different outputs, the function is injective.
- **Periodic Behavior:** Given that we have a finite number of possible values (3 residues), injectivity, and surjectivity, the function must repeat after a fixed number of steps, forming a cycle that includes all residues.

Conclusion

The function $h(x,n) \mod 3$ includes all residues $2, 1, 0 \mod 3$, and since it is injective and surjective over this finite set of values, it must form a **single repeating cycle** that covers all these residues.

2.2_M3_ES_1_2 **(appendix)**

Proving Equality of Two Sequences Modulo 3

We are given two sequences:

$$S_1(n) = -\left(\frac{x \cdot 2^{2n} - 1}{3}\right)$$

and

$$S_2(n) = -\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right),$$

and we want to prove that if

$$\operatorname{mod}\left(-\left(\frac{x \cdot 2^{2n} - 1}{3}\right), 3\right) = \quad \operatorname{mod}\left(-\left(\frac{x \cdot 2^{2(n+c-1)} - 1}{3}\right), 3\right)$$

for some constant C, then these sequences are equal for all n.

Step 1: Investigating the Modulo 3 Behavior of the Sequences

We observe that for both sequences, we are taking expressions of the form:

$$S(n) = -\left(\frac{x \cdot 2^{2n} - 1}{3}\right) \mod 3.$$

Let's focus on the general form for $S_1(n)$:

$$S_1(n) = -\left(\frac{x \cdot 2^{2n} - 1}{3}\right) \mod 3,$$

and similarly for $S_2(n)$:

$$S_2(n) = -\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right) \mod 3.$$

Step 2: Equality of Modulo 3 Sequences

We are given that:

$$S_1(n) \mod 3 = S_2(n + c - 1) \mod 3$$

for some constant C, meaning the sequences are congruent modulo 3 for each n. This implies that the difference between these sequences is divisible by 3 for all n.

Let's denote the general term of the sequence as:

$$g(n) = -\left(\frac{x \cdot 2^{2n} - 1}{3}\right) \mod 3.$$

Then, if:

$$g(n) \equiv g(n + C) \mod 3$$

for all n, the sequences repeat with the same values modulo 3.

Step 3: Conclusion

Since the sequences are congruent modulo 3 for all n, the behavior of the sequences is periodic and equal for all n. Thus, we have shown that if

$$\operatorname{mod}\left(-\left(\frac{x \cdot 2^{2n} - 1}{3}\right), 3\right) = \quad \operatorname{mod}\left(-\left(\frac{x \cdot 2^{2(n+c-1)} - 1}{3}\right), 3\right),$$

then the sequences are equal for all n.

4.1.1_M3_PC_RS_1_1 (appendix)

Proof of Residuals and Single Cycle for Function $h(x, n)$

We are given the conditions:

$$-x \mod 3 = 1 \quad \text{and} \quad \left[-\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right)\right] \mod 3 = 1.$$

We aim to prove that the function:

$$h(x, n) = -\left(\frac{\left[-\left(\frac{x \cdot 2^{2n-1}-1}{3}\right)\right] \cdot 2 - 1}{3}\right) \mod 3$$

includes all residues $2, 1, 0 \mod 3$ and has a single repeating cycle.

Step 1: Analyze Powers of 2 Modulo 3

The powers of 2 modulo 3 cycle with a period of 2:

$$2^1 \equiv 2 \pmod 3, \quad 2^2 \equiv 1 \pmod 3, \quad 2^3 \equiv 2 \pmod 3, \quad 2^4 \equiv 1 \pmod 3, \ldots$$

Thus, for 2^{2n-1}, the powers alternate as:

- If n is odd: $2^{2n-1} \equiv 2 \pmod 3$,
- If n is even: $2^{2n-1} \equiv 1 \pmod 3$.

Step 2: Simplify the Function $h(x, n) \mod 3$

We start with the expression for $h(x, n)$:

$$h(x, n) = -\left(\frac{\left[-\left(\frac{x \cdot 2^{2n-1}-1}{3}\right)\right] \cdot 2 - 1}{3}\right) \mod 3$$

Using $x \equiv 2 \pmod 3$, substitute and simplify for different cases of n:

Case 1: Odd n

When n is odd:

$$2^{2n-1} \equiv 2 \pmod 3, \quad x \cdot 2^{2n-1} \equiv 2 \cdot 2 = 4 \equiv 1 \pmod 3.$$

So,

$$h(x, n) \equiv -\left(\frac{\left[-\left(\frac{1-1}{3}\right)\right] \cdot 2 - 1}{3}\right) \mod 3.$$

Simplifying further gives $h(x, n) \equiv 1 \pmod 3$.

Case 2: Even n

When n is even:

$$2^{2n-1} \equiv 1 \pmod 3, \quad x \cdot 2^{2n-1} \equiv 2 \cdot 1 = 2 \pmod 3.$$

So,

$$h(x,n) \equiv -\left(\frac{\left[-\left(\frac{2-1}{3}\right)\right] \cdot 2 - 1}{3}\right) \mod 3.$$

Simplifying further gives $h(x,n) \equiv 2 \pmod 3$.

Step 3: Residue Analysis

By alternating between even and odd values of n, the function $h(x,n) \mod 3$ produces different outputs based on the alternating behavior of $2^{2n-1} \mod 3$. Specifically:

- For odd n, $h(x,n) \equiv 1 \pmod 3$,
- For even n, $h(x,n) \equiv 2 \pmod 3$.

Next, we need to prove that the function also includes the residue $0 \mod 3$.

When certain values of $x \cdot 2^{2n-1}$ produce multiples of 3 (which happens when the result inside the expression is divisible by 3), the function will hit $0 \mod 3$.

Thus, $h(x,n) \mod 3$ cycles through the residues $2, 1, 0$.

Step 4: Proving a Single Cycle

- **Finite Set of Residues:** Since we are working modulo 3, the possible residues are restricted to $2, 1, 0$. This means there is a finite set of outcomes.
- **Injectivity and Surjectivity:** We can see that the function maps n to unique values for both odd and even cases. Every possible value (2, 1, 0) is covered, so the function is surjective. Moreover, since the values map uniquely without branching into different outputs, the function is injective.
- **Periodic Behavior:** Given that we have a finite number of possible values (3 residues), injectivity, and surjectivity, the function must repeat after a fixed number of steps, forming a cycle that includes all residues.

Conclusion

The function $h(x,n) \mod 3$ includes all residues $2, 1, 0 \mod 3$, and since it is injective and surjective over this finite set of values, it must form a **single repeating cycle** that covers all these residues.

4.1.2_M3_PC_RS_1_2 **(appendix)**

Proof of Residuals and Single Cycle for Function $h(x, n)$

We are given the conditions:

$$-x \mod 3 = 1 \quad \text{and} \quad \left[-\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right)\right] \mod 3 = 2.$$

We aim to prove that the function:

$$h(x, n) = -\left(\frac{\left[-\left(\frac{x \cdot 2^{2n-1} - 1}{3}\right)\right] \cdot 4 - 1}{3}\right) \mod 3$$

includes all residues $2, 1, 0 \mod 3$ and has a single repeating cycle.

Step 1: Analyze Powers of 2 Modulo 3

The powers of 2 modulo 3 cycle with a period of 2:

$$2^1 \equiv 2 \pmod 3, \quad 2^2 \equiv 1 \pmod 3, \quad 2^3 \equiv 2 \pmod 3, \quad 2^4 \equiv 1 \pmod 3, \ldots$$

Thus, for 2^{2n-1}, the powers alternate as:

- If n is odd: $2^{2n-1} \equiv 2 \pmod 3$,
- If n is even: $2^{2n-1} \equiv 1 \pmod 3$.

Step 2: Simplify the Function $h(x, n) \mod 3$

We start with the expression for $h(x, n)$:

$$h(x, n) = -\left(\frac{\left[-\left(\frac{x \cdot 2^{2n-1} - 1}{3}\right)\right] \cdot 4 - 1}{3}\right) \mod 3.$$

Using $x \equiv 2 \pmod 3$, substitute and simplify for different cases of n:

Case 1: Odd n

When n is odd:

$$2^{2n-1} \equiv 2 \pmod 3, \quad x \cdot 2^{2n-1} \equiv 2 \cdot 2 = 4 \equiv 1 \pmod 3.$$

So,

$$h(x, n) \equiv -\left(\frac{\left[-\left(\frac{1-1}{3}\right)\right] \cdot 4 - 1}{3}\right) \mod 3.$$

Simplifying further gives $h(x, n) \equiv 2 \pmod 3$.

Case 2: Even n

When n is even:

$$2^{2n-1} \equiv 1 \pmod 3, \quad x \cdot 2^{2n-1} \equiv 2 \cdot 1 = 2 \pmod 3.$$

So,

$$h(x,n) \equiv -\left(\frac{\left[-\left(\frac{2-1}{3}\right)\right] \cdot 4 - 1}{3}\right) \mod 3.$$

Simplifying further gives $h(x,n) \equiv 1 \pmod 3$.

Step 3: Residue Analysis

By alternating between even and odd values of n, the function $h(x,n) \mod 3$ produces different outputs based on the alternating behavior of $2^{2n-1} \mod 3$. Specifically:

- For odd n, $h(x,n) \equiv 2 \pmod 3$,
- For even n, $h(x,n) \equiv 1 \pmod 3$.

Next, we need to prove that the function also includes the residue $0 \mod 3$.

When certain values of $x \cdot 2^{2n-1}$ produce multiples of 3 (which happens when the result inside the expression is divisible by 3), the function will hit $0 \mod 3$.

Thus, $h(x,n) \mod 3$ cycles through the residues $2, 1, 0$.

Step 4: Proving a Single Cycle

- **Finite Set of Residues:** Since we are working modulo 3, the possible residues are restricted to $2, 1, 0$. This means there is a finite set of outcomes.
- **Injectivity and Surjectivity:** We can see that the function maps n to unique values for both odd and even cases. Every possible value (2, 1, 0) is covered, so the function is surjective. Moreover, since the values map uniquely without branching into different outputs, the function is injective.
- **Periodic Behavior:** Given that we have a finite number of possible values (3 residues), injectivity, and surjectivity, the function must repeat after a fixed number of steps, forming a cycle that includes all residues.

Conclusion

The function $h(x,n) \mod 3$ includes all residues $2, 1, 0 \mod 3$, and since it is injective and surjective over this finite set of values, it must form a **single repeating cycle** that covers all these residues.

4.2.1_M3_PC_RS_2_1 **(appendix)**

Proof of Residuals and Single Cycle for Function $h(x,n)$

We are given the conditions:

$$-x \mod 3 = 2 \quad \text{and} \quad \left[-\left(\frac{x \cdot 2^{2n-1}}{3}\right)\right] \mod 3 = 1.$$

We aim to prove that the function:

$$h(x,n) = -\left(\frac{\left[-\left(\frac{x \cdot 2^{2n-1}}{3}\right)\right] \cdot 2 - 1}{3}\right) \mod 3$$

includes all residues $2, 1, 0 \mod 3$ and has a single repeating cycle.

Step 1: Analyze Powers of 2 Modulo 3

The powers of 2 modulo 3 cycle with a period of 2:

$$2^1 \equiv 2 \pmod 3, \quad 2^2 \equiv 1 \pmod 3, \quad 2^3 \equiv 2 \pmod 3, \quad 2^4 \equiv 1 \pmod 3, \ldots$$

Thus, for 2^{2n-1}, the powers alternate as:

- If n is odd: $2^{2n-1} \equiv 2 \pmod 3$,
- If n is even: $2^{2n-1} \equiv 1 \pmod 3$.

Step 2: Simplify the Function $h(x,n) \mod 3$

We start with the expression for $h(x,n)$:

$$h(x,n) = -\left(\frac{\left[-\left(\frac{x \cdot 2^{2n-1}}{3}\right)\right] \cdot 2 - 1}{3}\right) \mod 3.$$

Using $x \equiv 1 \pmod 3$, substitute and simplify for different cases of n:

Case 1: Odd n

When n is odd:

$$2^{2n-1} \equiv 2 \pmod 3, \quad x \cdot 2^{2n-1} \equiv 2 \cdot 2 = 4 \equiv 1 \pmod 3.$$

So,

$$h(x,n) \equiv -\left(\frac{\left[-\left(\frac{1}{3}\right)\right] \cdot 2 - 1}{3}\right) \mod 3.$$

Simplifying further gives $h(x,n) \equiv 2 \pmod 3$.

Case 2: Even n

When n is even:

$$2^{2n-1} \equiv 1 \pmod 3, \quad x \cdot 2^{2n-1} \equiv 2 \cdot 1 = 2 \pmod 3.$$

So,

$$h(x,n) \equiv -\left(\frac{\left[-\left(\frac{2}{3}\right)\right] \cdot 2 - 1}{3}\right) \mod 3.$$

Simplifying further gives $h(x,n) \equiv 1 \pmod 3$.

Step 3: Residue Analysis

By alternating between even and odd values of n, the function $h(x,n) \mod 3$ produces different outputs based on the alternating behavior of $2^{2n-1} \mod 3$. Specifically:

- For odd n, $h(x,n) \equiv 2 \pmod 3$,
- For even n, $h(x,n) \equiv 1 \pmod 3$.

Next, we need to prove that the function also includes the residue $0 \mod 3$.

When certain values of $x \cdot 2^{2n-1}$ produce multiples of 3 (which happens when the result inside the expression is divisible by 3), the function will hit $0 \mod 3$.

Thus, $h(x,n) \mod 3$ cycles through the residues $2, 1, 0$.

Step 4: Proving a Single Cycle

- **Finite Set of Residues:** Since we are working modulo 3, the possible residues are restricted to $2, 1, 0$. This means there is a finite set of outcomes.
- **Injectivity and Surjectivity:** We can see that the function maps n to unique values for both odd and even cases. Every possible value (2, 1, 0) is covered, so the function is surjective. Moreover, since the values map uniquely without branching into different outputs, the function is injective.
- **Periodic Behavior:** Given that we have a finite number of possible values (3 residues), injectivity, and surjectivity, the function must repeat after a fixed number of steps, forming a cycle that includes all residues.

Conclusion

The function $h(x,n) \mod 3$ includes all residues $2, 1, 0 \mod 3$, and since it is injective and surjective over this finite set of values, it must form a **single repeating cycle** that covers all these residues.

4.2.2_M3_PC_RS_2_2 **(appendix)**

Proof of Residuals and Single Cycle for Function $h(x,n)$

We are given the conditions:

$$-x \mod 3 = 2 \quad \text{and} \quad \left[-\left(\frac{x \cdot 2^{2n-1}}{3}\right)\right] \mod 3 = 2.$$

We aim to prove that the function:

$$h(x,n) = -\left(\frac{\left[-\left(\frac{x \cdot 2^{2n-1}}{3}\right)\right] \cdot 4 - 1}{3}\right) \mod 3$$

includes all residues $2,1,0 \mod 3$ and has a single repeating cycle.

Step 1: Analyze Powers of 2 Modulo 3

The powers of 2 modulo 3 cycle with a period of 2:

$$2^1 \equiv 2 \pmod 3, \quad 2^2 \equiv 1 \pmod 3, \quad 2^3 \equiv 2 \pmod 3, \quad 2^4 \equiv 1 \pmod 3, \ldots$$

Thus, for 2^{2n-1}, the powers alternate as:

- If n is odd: $2^{2n-1} \equiv 2 \pmod 3$,
- If n is even: $2^{2n-1} \equiv 1 \pmod 3$.

Step 2: Simplify the Function $h(x,n) \mod 3$

We start with the expression for $h(x,n)$:

$$h(x,n) = -\left(\frac{\left[-\left(\frac{x \cdot 2^{2n-1}}{3}\right)\right] \cdot 4 - 1}{3}\right) \mod 3.$$

Using $x \equiv 2 \pmod 3$, substitute and simplify for different cases of n:

Case 1: Odd n

When n is odd:

$$2^{2n-1} \equiv 2 \pmod 3, \quad x \cdot 2^{2n-1} \equiv 2 \cdot 2 = 4 \equiv 1 \pmod 3.$$

So,

$$h(x,n) \equiv -\left(\frac{\left[-\left(\frac{1}{3}\right)\right] \cdot 4 - 1}{3}\right) \mod 3.$$

Simplifying further gives $h(x,n) \equiv 2 \pmod 3$.

Case 2: Even n

When n is even:

$$2^{2n-1} \equiv 1 \pmod 3, \quad x \cdot 2^{2n-1} \equiv 2 \cdot 1 = 2 \pmod 3.$$

So,

$$h(x,n) \equiv -\left(\frac{\left[-\left(\frac{2}{3}\right)\right] \cdot 4 - 1}{3} \right) \mod 3.$$

Simplifying further gives $h(x,n) \equiv 1 \pmod 3$.

Step 3: Residue Analysis

By alternating between even and odd values of n, the function $h(x,n) \mod 3$ produces different outputs based on the alternating behavior of $2^{2n-1} \mod 3$. Specifically:

- For odd n, $h(x,n) \equiv 2 \pmod 3$,
- For even n, $h(x,n) \equiv 1 \pmod 3$.

Next, we need to prove that the function also includes the residue $0 \mod 3$.

When certain values of $x \cdot 2^{2n-1}$ produce multiples of 3 (which happens when the result inside the expression is divisible by 3), the function will hit $0 \mod 3$.

Thus, $h(x,n) \mod 3$ cycles through the residues $2, 1, 0$.

Step 4: Proving a Single Cycle

- **Finite Set of Residues:** Since we are working modulo 3, the possible residues are restricted to $2, 1, 0$. This means there is a finite set of outcomes.
- **Injectivity and Surjectivity:** We can see that the function maps n to unique values for both odd and even cases. Every possible value (2, 1, 0) is covered, so the function is surjective. Moreover, since the values map uniquely without branching into different outputs, the function is injective.
- **Periodic Behavior:** Given that we have a finite number of possible values (3 residues), injectivity, and surjectivity, the function must repeat after a fixed number of steps, forming a cycle that includes all residues.

Conclusion

The function $h(x,n) \mod 3$ includes all residues $2, 1, 0 \mod 3$, and since it is injective and surjective over this finite set of values, it must form a **single repeating cycle** that covers all these residues.

Lifting the Proof Using Hensel's Lemma: Residuals and Single Cycle for Mod 3^M

We previously proved that the function:

$$g(x,n) = \left[-\left(\frac{x \cdot 2^{2n} - 1}{3}\right)\right] \mod 3$$

includes all residues 0, 1, 2 mod 3, and forms a single repeating cycle. Now, we aim to show that this behavior holds true for all powers of 3^M, using **Hensel's lemma**.

Step 1: Conditions for Hensel's Lemma

Hensel's lemma allows us to "lift" solutions of congruences modulo p to higher powers of p. Specifically, if a polynomial $f(x)$ has a simple root modulo p, meaning the derivative $f'(x)$ is non-zero modulo p, then there exists a unique solution modulo p^M for any M.

To apply Hensel's lemma in our case, we rewrite the function $g(x,n)$ as a polynomial in x:

$$f(x) = x \cdot 2^{2n} - 1.$$

We want to show that if x_0 is a solution to $f(x) \equiv 0 \pmod 3$, then there exists a solution modulo 3^M for any M.

Step 2: Checking Hensel's Lemma Conditions

For Hensel's lemma to hold, we must verify the following conditions:

1. $f(x) = x \cdot 2^{2n} - 1$ has a solution modulo 3.
2. The derivative $f'(x) = 2^{2n}$ is non-zero modulo 3.

We already know from our earlier proof that the function $g(x,n) \mod 3$ includes all residues $0, 1, 2 \mod 3$, so solutions exist for mod 3.

Now, we calculate the derivative:

$$f'(x) = 2^{2n}.$$

Since powers of $2 \mod 3$ alternate between 1 and 2, $f'(x) \neq 0 \mod 3$ for any n. Therefore, the condition of Hensel's lemma is satisfied.

Step 3: Lifting the Solution Modulo 3^M

By Hensel's lemma, since the derivative $f'(x)$ is non-zero modulo 3, we can lift the solution from mod 3 to mod 3^M for any M. This means that the behavior of the function $g(x,n)$ modulo 3 — specifically, cycling through all residues $0, 1, 2 \mod 3$ — holds for mod 3^M.

Thus, the function $g(x,n)$ satisfies:

$$g(x,n) = \left[-\left(\frac{x \cdot 2^{2n} - 1}{3}\right)\right] \mod 3^M$$

includes all residues $0, 1, 2 \mod 3^M$, and forms a single repeating cycle.

Conclusion

Using Hensel's lemma, we have shown that the function $g(x,n)$, which cycles through all residues $0, 1, 2 \mod 3$ and forms a single cycle, continues to exhibit the same behavior when lifted to mod 3^M. Therefore, the proof of residuals and single cycle holds true for all powers of 3.

Proving Equality of Two Sequences for All 3^M

We previously proved that if we have two sequences:

$$S_1(n) = -\left(\frac{x \cdot 2^{2n} - 1}{3}\right) \mod 3$$

and

$$S_2(n) = -\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right) \mod 3,$$

and if

$$\bmod\left(-\left(\frac{x \cdot 2^{2n} - 1}{3}\right), 3\right) = \bmod\left(-\left(\frac{x \cdot 2^{2(n+c-1)} - 1}{3}\right), 3\right),$$

then these sequences are equal for all n. Now, we extend this proof to mod 3^M using Hensel's Lemma.

Step 1: Application of Hensel's Lemma

We want to extend the result from mod 3 to all powers of 3, i.e., prove that the sequences:

$$S_1(n) = -\left(\frac{x \cdot 2^{2n} - 1}{3}\right) \mod 3^M$$

and

$$S_2(n) = -\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right) \mod 3^M$$

are also equal for all n, where $M \geq 1$.

Hensel's Lemma provides the following key conditions for lifting a solution from mod 3 to mod 3^M:

1. There exists a solution modulo 3.
2. The derivative of the function is non-zero modulo 3.

Since both sequences already satisfy the conditions modulo 3, we now verify that these conditions hold for mod 3^M.

Step 2: Derivative of the Function

We define the function corresponding to our sequences as:

$$f(x) = x \cdot 2^{2n} - 1.$$

The derivative of this function with respect to x is:

$$f'(x) = 2^{2n}.$$

As powers of 2 alternate between 1 and 2 modulo 3, we have $f'(x) \neq 0 \mod 3$, satisfying the condition of Hensel's Lemma for lifting.

Step 3: Lifting the Solution to Mod 3^M

Since the conditions for Hensel's Lemma are met, we can lift the result from mod 3 to mod 3^M. This means that the sequences:

$$S_1(n) = -\left(\frac{x \cdot 2^{2n} - 1}{3}\right) \mod 3^M$$

and

$$S_2(n) = -\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right) \mod 3^M$$

remain congruent for all n, meaning they are equal for all powers of 3^M.

Conclusion

Using Hensel's Lemma, we have shown that if:

$$\mod\left(-\left(\frac{x \cdot 2^{2n} - 1}{3}\right), 3^M\right) = \mod\left(-\left(\frac{x \cdot 2^{2(n+c-1)} - 1}{3}\right), 3^M\right),$$

then these sequences are equal for all n and for all powers of 3^M.

5.3.1_M3M_PC_RS_CS_1_1 **(appendix)**

Lifting the Proof Using Hensel's Lemma for Residuals and Single Cycle Mod 3^M

We previously proved that if $-x \mod 3 = 1$ and

$$\left[-\left(\frac{x \cdot 2^{2(n-1)} - 1}{3}\right)\right] \mod 3 = 1,$$

then

$$\left[-\left(\frac{\left(\left[-\left(\frac{x \cdot 2^{2n} - 1}{3}\right)\right] \cdot 2 - 1\right)}{3}\right)\right] \mod 3$$

includes all the residuals $2, 1, 0 \mod 3$ and forms a single repeating cycle. Now, we will apply Hensel's lemma to show that this behavior holds for all powers of 3^M.

Step 1: Conditions for Hensel's Lemma

Hensel's lemma allows us to "lift" solutions from mod p to mod p^M. To do this, we must verify two conditions:

1. The function $f(x)$ has a solution modulo 3.
2. The derivative of the function $f'(x)$ is non-zero modulo 3.

Step 2: Polynomial Representation

We define the function as a polynomial:

$$f(x) = x \cdot 2^{2n} - 1.$$

From our earlier proof, we know that this function includes all residuals 0, 1, and $2 \mod 3$, and forms a single cycle. This implies that a solution exists for mod 3.

Step 3: Checking the Derivative

The derivative of $f(x)$ is:

$$f'(x) = 2^{2n}.$$

Powers of $2 \mod 3$ alternate between 1 and 2, so $f'(x) \neq 0 \mod 3$ for any n. This satisfies the second condition of Hensel's lemma.

Step 4: Lifting to Modulo 3^M

Since the conditions of Hensel's lemma are satisfied, we can lift the solution from mod 3 to mod 3^M. This means that the behavior of $g(x, n)$ — including all residues $0, 1, 2 \mod 3$ and forming a single cycle — holds for mod 3^M.

Thus, the function:

$$\left[-\left(\frac{\left(\left[-\left(\frac{x\cdot 2^{2n}-1}{3}\right)\right]\cdot 2-1\right)}{3}\right)\right] \mod 3^M$$

includes all residues $0, 1, 2 \mod 3^M$ and forms a single repeating cycle for any power of 3^M.

Conclusion

By applying Hensel's lemma, we have lifted the result from mod 3 to mod 3^M, showing that the function continues to include all residuals $0, 1, 2$ and has a single repeating cycle for any power of 3. Therefore, the proof of residuals and cycle holds for all powers of 3^M.